

INTRINSIC THEORY OF PROJECTIVE CHANGES IN FINSLER GEOMETRY

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Abstract. The aim of the present paper is to provide an *intrinsic* investigation of projective changes in Finsler geometry, following the pullback formalism. Various known local results are generalized and other new intrinsic results are obtained. Nontrivial characterizations of projective changes are given. The fundamental projectively invariant tensors, namely, the projective deviation tensor, the Weyl torsion tensor, the Weyl curvature tensor and the Douglas tensor are investigated. The properties of these tensors and their interrelationships are obtained. Projective connections and projectively flat manifolds are characterized. The present work is entirely intrinsic (free from local coordinates).

Keywords. Pullback formalism, Projective change, Canonical spray, Barthel connection, Berwald connection, Weyl curvature tensor, Weyl torsion tensor, Douglas tensor, Projective connection, Projectively flat manifold.

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Introduction

The most well-known and widely used approaches to *global* Finsler geometry are the Klein-Grifone (KG-) approach ([6], [7], [8]) and the pullback (PB-) approach ([1], [13], [17]). The universe of the first approach is the tangent bundle of TM , whereas the universe of the second is the pullback of the tangent bundle TM by $\pi : TM \rightarrow M$. Each of the two approaches has its own geometry which differs significantly from the geometry of the other (in spite of the existence of some links between them).

The theory of projective changes in Riemannian geometry has been deeply studied (*locally and intrinsically*) by many authors. With regard to Finsler geometry, a complete *local* theory of projective changes has been established ([4], [9], [10], [11], [14], \dots). Moreover, an intrinsic theory of projective changes (resp. semi-projective changes) has been investigated in [3], [12] (resp. [16]) following the KG-approach. To the best of our knowledge, there is no complete intrinsic theory in the PB-approach.

In this paper we present *an intrinsic* theory of projective changes in Finsler geometry following *the pullback approach*. Various known local results are generalized and other new intrinsic results are obtained.

The paper consists of four parts preceded by an introductory section (§1), which provides a brief account of the basic definitions and concepts necessary for this work.

In the first part (§2), the projective change of Barthel and Berwald connections, as well as their curvatur tensors, are investigated. Some characterizations of projective changes are established. The results obtained in this section play a key role in obtaining other results in the next sections.

The second part (§3) is devoted to an investigation of the fundamental projectively invariant tensors under a projective change, namely, the projective deviation tensor, the Weyl torsion tensor and the Weyl curvature tensor. The properties of these tensors and their interrelationships are studied.

The third part (§4) provides a characterization of a linear connection which is invariant under projective changes (the projective connection). Moreover, another fundamental projectively invariant tensor (the Douglas tensor) is investigated, the properties of which are discussed. Finally, the Douglas tensor and the projective connection are related.

In the fourth and last part (§5), the projective change of some important special Finsler manifolds, namely, the Berwald, Douglas and projectively flat Finsler manifolds are investigated. Moreover, the relationship between projectively flat Finsler manifolds and Douglas tensor (Weyl tensor) is obtained.

It should finally be noted that the present work is entirely intrinsic (free from local coordinates).

1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1] and [13]. We make the assumption that the geometric objects we consider are of class C^∞ . The following notation will be used throughout this paper:

M : a real paracompact differentiable manifold of finite dimension n and of class C^∞ ,
 $\mathfrak{F}(M)$: the \mathbb{R} -algebra of differentiable functions on M ,

$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$ -module of vector fields on M ,
 $\pi_M : TM \longrightarrow M$: the tangent bundle of M ,
 $\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M ,
 $V(TM)$: the vertical subbundle of the bundle $T(TM)$,
 $P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$: the pullback of the tangent bundle TM by π ,
 $\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(\mathcal{T}M)$ -module of differentiable sections of $\pi^{-1}(TM)$,
 i_X : the interior product with respect to $X \in \mathfrak{X}(M)$,
 df : the exterior derivative of $f \in \mathfrak{F}(M)$,
 $d_L := [i_L, d]$, i_L being the interior derivative with respect to a vector form L .

Elements of $\mathfrak{X}(\pi(M))$ will be called π -vector fields and will be denoted by barred letters \bar{X} . Tensor fields on $\pi^{-1}(TM)$ will be called π -tensor fields. The fundamental π -vector field is the π -vector field $\bar{\eta}$ defined by $\bar{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$. We have the following short exact sequence of vector bundles:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

the bundle morphisms ρ and γ being defined in [17]. The vector 1-form $J := \gamma \circ \rho$ is called the natural almost tangent structure of TM . The vertical vector field $\mathcal{C} := \gamma \circ \bar{\eta}$ on TM is called the canonical (Liouville) vector field.

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. The connection (or the deflection) map associated with D is defined by

$$K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X \bar{\eta}.$$

A tangent vector $X \in T_u(\mathcal{T}M)$ at $u \in \mathcal{T}M$ is horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ is called the horizontal space at u . The connection D is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \quad \forall u \in \mathcal{T}M.$$

Let $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$, called the horizontal map of the connection D , then

$$\rho \circ \beta = id_{\pi^{-1}(TM)}, \quad \beta \circ \rho = id_{H(\mathcal{T}M)} \text{ on } H(\mathcal{T}M).$$

For a regular connection D , the horizontal and vertical covariant derivatives $\overset{1}{D}$ and $\overset{2}{D}$ are defined, for a vector (1) π -form A , for example, by

$$(\overset{1}{D} A)(\bar{X}, \bar{Y}) := (D_{\beta \bar{X}} A)(\bar{Y}), \quad (\overset{2}{D} A)(\bar{X}, \bar{Y}) := (D_{\gamma \bar{X}} A)(\bar{Y}).$$

The (classical) torsion tensor \mathbf{T} of the connection D is given by

$$\mathbf{T}(X, Y) = D_X \rho Y - D_Y \rho X - \rho[X, Y] \quad \forall X, Y \in \mathfrak{X}(\mathcal{T}M),$$

from which the horizontal ((h)h-) and mixed ((h)hv-) torsion tensors are defined respectively by

$$Q(\bar{X}, \bar{Y}) := \mathbf{T}(\beta \bar{X} \beta \bar{Y}), \quad T(\bar{X}, \bar{Y}) := \mathbf{T}(\gamma \bar{X}, \beta \bar{Y}) \quad \forall \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M)).$$

The (classical) curvature tensor \mathbf{K} of the connection D is given by

$$\mathbf{K}(X, Y) \rho Z = -D_X D_Y \rho Z + D_Y D_X \rho Z + D_{[X, Y]} \rho Z \quad \forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M),$$

from which the horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors are defined respectively by

$$R(\bar{X}, \bar{Y})\bar{Z} := \mathbf{K}(\beta\bar{X}\beta\bar{Y})\bar{Z}, \quad P(\bar{X}, \bar{Y})\bar{Z} := \mathbf{K}(\beta\bar{X}, \gamma\bar{Y})\bar{Z}, \quad S(\bar{X}, \bar{Y})\bar{Z} := \mathbf{K}(\gamma\bar{X}, \gamma\bar{Y})\bar{Z}.$$

The (v)h-, (v)hv- and (v)v-torsion tensors are defined respectively by

$$\hat{R}(\bar{X}, \bar{Y}) := R(\bar{X}, \bar{Y})\bar{\eta}, \quad \hat{P}(\bar{X}, \bar{Y}) := P(\bar{X}, \bar{Y})\bar{\eta}, \quad \hat{S}(\bar{X}, \bar{Y}) := S(\bar{X}, \bar{Y})\bar{\eta}.$$

For a Finsler manifold (M, L) , there are canonically associated two fundamental linear connections, namely, the Cartan connection ∇ and the Berwald connection D° . An explicit expression for the Berwald connection is given by

Theorem 1.1. [17] *The Berwald connection D° is uniquely determined by*

$$(a) \quad D_{\gamma\bar{X}}^\circ\bar{Y} = \rho[\gamma\bar{X}, \beta\bar{Y}], \quad (b) \quad D_{\beta\bar{X}}^\circ\bar{Y} = K[\beta\bar{X}, \gamma\bar{Y}].$$

We terminate this section by some concepts and results concerning the Klein-Grifone approach to intrinsic Finsler geometry. For more details, we refer to [6], [7] and [8].

Proposition 1.2. [8] *Let (M, L) be a Finsler manifold. The vector field G on TM defined by $i_G\Omega = -dE$ is a spray, where $E := \frac{1}{2}L^2$ is the energy function and $\Omega := dd_JE$. Such a spray is called the canonical spray.*

A nonlinear connection on M is a vector 1-form Γ on TM , C^∞ on TM , such that $J\Gamma = J$, $\Gamma J = -J$. The horizontal and vertical projectors associated with Γ are defined by $h := \frac{1}{2}(I + \Gamma)$ and $v := \frac{1}{2}(I - \Gamma)$ respectively. The torsion and curvature of Γ are defined by $t := \frac{1}{2}[J, \Gamma]$ and $\mathfrak{R} := -\frac{1}{2}[h, h]$ respectively. A nonlinear connection Γ is conservative if $d_h E = 0$.

Theorem 1.3. [7] *On a Finsler manifold (M, L) , there exists a unique conservative homogenous nonlinear connection with zero torsion. It is given by: $\Gamma = [J, G]$.*

Such a nonlinear connection is called the canonical connection, the Barthel connection or the Cartan nonlinear connection associated with (M, L) .

2. Projective changes and Berwald connection

In this section, the projective change of Barthel and Berwald connections, as well as their curvature tensors, are investigated. Some characterizations of projective changes are established.

Definition 2.1. *Let (M, L) be a Finsler manifold. A change $L \longrightarrow \tilde{L}$ of Finsler structures is said to be a projective change if each geodesic of (M, L) is a geodesic of (M, \tilde{L}) and vice versa. In this case, the Finsler manifolds (M, L) and (M, \tilde{L}) are said to be projectively related.*

Let $c : I \longrightarrow M : s \longmapsto (x^i(s)); i = 1, 2, \dots, n$, be a regular curve on a Finsler manifold (M, L) . If G is the canonical spray associated with (M, L) , then c is a geodesic

iff $G(c') = c''$, where $c' = \frac{dc}{ds}$ (s being the arc-length). In local coordinates, c is a geodesic iff

$$(x^i)'' + 2G^i(x, x') = 0.$$

If this equation is subjected to an arbitrary transformation of its parameter $s \mapsto t = t(s)$, with $\frac{dt}{ds} \neq 0$, then c is a geodesic iff

$$y^i = \frac{dx^i}{dt}, \quad \frac{dy^i}{dt} + 2G^i(x, y) = \mu y^i,$$

where $\mu := -(\frac{d^2t}{ds^2})/(\frac{dt}{ds})^2$. It is clear that these equations remain unchanged if we replace the functions $G^i(x, y)$ by new functions $\widetilde{G}^i(x, y)$, the latter being defined by

$$\widetilde{G}^i(x, y) = G^i(x, y) + \lambda(x, y)y^i, \quad (2.1)$$

where $\lambda(x, y)$ is an arbitrary function which is positively homogenous of degree 1 in the directional argument y . This result can be formulated intrinsically as follows.

Theorem 2.2. *Two Finsler manifolds (M, L) and (M, \widetilde{L}) are projectively related if, and only if, the associated canonical sprays G and \widetilde{G} are related by*

$$\widetilde{G} = G - 2\lambda(x, y)\mathcal{C}, \quad (2.2)$$

where $\lambda(x, y)$ is a function on TM , positively homogenous of degree 1 in y .

Remark 2.3. *It is to be noted that the local expression of (2.2) reduces to (2.1) and is in accordance with the existing classical local results on projective changes [11], [14]. For this reason, we have inserted the factor (-2) in (2.2). Moreover, this factor facilitates calculations.*

In what follows and throughout we will take (2.2) as the definition of a projective change.

The following lemma is useful for subsequence use.

Lemma 2.4. *The π -form α defined by*

$$\alpha(\overline{X}) := d_J\lambda(\beta\overline{X}) \quad (2.3)$$

has the following properties:

$$(a) \ (D_{\gamma\overline{X}}^\circ\alpha)(\overline{Y}) = dd_J\lambda(\gamma\overline{X}, \beta\overline{Y}), \quad (b) \ D_{\gamma\overline{\eta}}^\circ\alpha = 0, \quad (c) \ \alpha(\overline{\eta}) = \lambda.$$

Proof.

(a) We use Theorem 1.1 and the identities $\beta \circ \rho + \gamma \circ K = I$ and $i_J d_J \lambda = 0$:

$$\begin{aligned} dd_J\lambda(\gamma\overline{X}, \beta\overline{Y}) &= \gamma\overline{X} \cdot d_J\lambda(\beta\overline{Y}) - d_J\lambda([\gamma\overline{X}, \beta\overline{Y}]) \\ &= \gamma\overline{X} \cdot \alpha(\overline{Y}) - d_J\lambda(\beta\rho[\gamma\overline{X}, \beta\overline{Y}]) \\ &= \gamma\overline{X} \cdot \alpha(\overline{Y}) - \alpha(D_{\gamma\overline{X}}^\circ\overline{Y}) = (D_{\gamma\overline{X}}^\circ\alpha)(\overline{Y}). \end{aligned}$$

(b) From (a) above, we have

$$\begin{aligned} (D_{\gamma\overline{\eta}}^\circ\alpha)(\overline{X}) &= \mathcal{C} \cdot d_J\lambda(\beta\overline{X}) - d_J\lambda([\mathcal{C}, \beta\overline{X}]) = \mathcal{C} \cdot (\gamma\overline{X} \cdot \lambda) - J[JG, \gamma\overline{X}] \cdot \lambda \\ &= \mathcal{C} \cdot (\gamma\overline{X} \cdot \lambda) - \{[JG, \gamma\overline{X}] - J[G, \gamma\overline{X}]\} \cdot \lambda, \quad (as \ J^2 = 0) \\ &= \{\mathcal{C} \cdot (\gamma\overline{X} \cdot \lambda) - [JG, \gamma\overline{X}] \cdot \lambda\} - JX \cdot \lambda = \gamma\overline{X} \cdot (\mathcal{C} \cdot \lambda) - \gamma\overline{X} \cdot \lambda. \end{aligned}$$

As $\mathcal{C} \cdot \lambda = \lambda$, the result follows. \square

An important characterization of projective changes is given by

Theorem 2.5. *The following assertions are equivalent:*

(a) (M, L) and (M, \tilde{L}) are projectively related.

(b) The associated Barthel connections Γ and $\tilde{\Gamma}$ are related by

$$\tilde{\Gamma} = \Gamma - 2\{\lambda J + d_J \lambda \otimes \mathcal{C}\}.$$

(c) The associated Berwald connections D° and \tilde{D}° are related by

$$\tilde{D}_X^\circ \bar{Y} = D_X^\circ \bar{Y} + \alpha(\bar{Y})\rho X + \alpha(\rho X)\bar{Y} + (D^\circ \alpha)(\bar{Y}, \rho X)\bar{\eta}. \quad (2.4)$$

Proof.

(a) \implies (b): As $\tilde{G} = G - 2\lambda(x, y)\mathcal{C}$, we have

$$\tilde{\Gamma} = [J, \tilde{G}] = [J, G - 2\lambda\mathcal{C}] = [J, G] - 2[J, \lambda\mathcal{C}].$$

Since, for every vector form A on TM , $X \in \mathfrak{X}(TM)$ and $f \in \mathfrak{F}(TM)$ [5],

$$[A, fX] = f[A, X] + d_A f \otimes X - df \wedge i_X A,$$

we obtain

$$\begin{aligned} [J, \lambda\mathcal{C}] &= \lambda[J, \mathcal{C}] + d_J \lambda \otimes \mathcal{C} - d\lambda \wedge i_{\mathcal{C}} J \\ &= \lambda J + d_J \lambda \otimes \mathcal{C}, \quad (\text{as } [J, \mathcal{C}] = J \text{ and } i_{\mathcal{C}} J = 0). \end{aligned}$$

From which (b) follows.

(b) \implies (c): If (b) holds, then

$$\tilde{h} = h - \lambda J - d_J \lambda \otimes \mathcal{C}, \quad \tilde{v} = v + \lambda J + d_J \lambda \otimes \mathcal{C}. \quad (2.5)$$

Using Theorem 1.1 and (2.5), we get

$$\begin{aligned} \tilde{D}_{\tilde{v}X}^\circ \rho Y &= \rho[\tilde{v}X, Y] = \rho[vX, Y] + \rho[\lambda JX + d_J \lambda(X)\mathcal{C}, Y] \\ &= \rho[vX, Y] + \{\lambda\rho[JX, Y] - (Y \cdot \lambda)\rho JX\} \\ &\quad + \{d_J \lambda(X)\rho[\mathcal{C}, Y] - (Y \cdot d_J \lambda(X))\rho(\mathcal{C})\} \\ &= D_{vX}^\circ \rho Y + \lambda\rho[JX, Y] + \alpha(\rho X)\rho[\mathcal{C}, Y]. \end{aligned} \quad (2.6)$$

Similarly, one can show that

$$\begin{aligned} \gamma \tilde{D}_{\tilde{h}X}^\circ \rho Y &= \gamma D_{hX}^\circ \rho Y - \lambda v \{J[X, JY] + J[JX, Y]\} \\ &\quad + (JY \cdot \lambda)v(JX) - d_J \lambda(X) \{J[\mathcal{C}, Y] + J[G, JY]\} \\ &\quad + (JY \cdot d_J \lambda(X))v(\mathcal{C}) + \lambda J[hX, JY] - (d_J \lambda([JY, hX]))\mathcal{C}. \end{aligned}$$

From which, taking into account the fact that $Jv = 0$, $vJ = J$ and $\gamma : \pi^{-1}(TM) \longrightarrow V(TM)$ is an isomorphism, we get

$$\begin{aligned} \tilde{D}_{\tilde{h}X}^\circ \rho Y &= D_{hX}^\circ \rho Y + d_J \lambda(Y)\rho X + d_J \lambda(X)\rho Y \\ &\quad + dd_J \lambda(JY, hX)\bar{\eta} - \lambda\rho[JX, Y] - d_J \lambda(X)\rho[\mathcal{C}, Y] \\ &= D_{hX}^\circ \rho Y + \alpha(\rho Y)\rho X + \alpha(\rho X)\rho Y + (D_{JY}^\circ \alpha)(\rho X)\bar{\eta} \\ &\quad - \lambda\rho[JX, Y] - \alpha(\rho X)\rho[\mathcal{C}, Y]. \end{aligned} \quad (2.7)$$

Hence, (2.4) follows from (2.6) and (2.7).

(c) \implies (a): Assume that Equation (2.4) holds. Then, by setting $\bar{Y} = \bar{\eta}$ in (2.4), noting that $\alpha(\bar{\eta}) = \lambda$ (Lemma 2.4(c)), we get

$$\tilde{K}^\circ(X) = K^\circ(X) + \lambda\rho X + \alpha(\rho X)\bar{\eta} + (D_{\gamma\bar{\eta}}^\circ\alpha)(\rho X)\bar{\eta}.$$

From which, together with Lemma 2.4(b) and the fact that $v = v^\circ = \gamma \circ K^\circ = \gamma \circ K$ [17], we get

$$\tilde{v}X = vX + \lambda JX + \alpha(\rho X)\mathcal{C}.$$

Consequently,

$$\tilde{h}X = hX - \lambda JX - \alpha(\rho X)\mathcal{C}.$$

Setting $X = G$ in the last relation, taking into account Lemma 2.4(c) and $\tilde{h}G = \tilde{\beta}\bar{\eta} = \tilde{G}$, we obtain

$$\tilde{G} = G - 2\lambda\mathcal{C}.$$

Hence, by Theorem 2.2, (M, L) and (M, \tilde{L}) are projectively related. \square

Corollary 2.6. *Under the projective change (2.2), the curvature tensors \mathfrak{R} and $\tilde{\mathfrak{R}}$ of the associated Barthel connections Γ and $\tilde{\Gamma}$ are related by*

$$\tilde{\mathfrak{R}} = \mathfrak{R} + \frac{1}{2} (2d_h\lambda - d_J\lambda^2) \wedge J + d_h d_J \lambda \otimes \mathcal{C}.$$

Corollary 2.7. *In view of Theorem 2.5(c), we have*

(a) $\tilde{D}_{\gamma\bar{X}}^\circ\bar{Y} = D_{\gamma\bar{X}}^\circ\bar{Y}.$

(b) $\tilde{D}_{\beta\bar{X}}^\circ\bar{Y} = D_{\beta\bar{X}}^\circ\bar{Y} + \alpha(\bar{Y})\bar{X} + \alpha(\bar{X})\bar{Y} + D_{\beta\bar{X}}^\circ\alpha(\bar{Y}, \rho X)\bar{\eta} - \lambda D_{\gamma\bar{X}}^\circ\bar{Y} - \alpha(\bar{X})D_{\gamma\bar{\eta}}^\circ\bar{Y}.$

Consequently,

(c) *The map $D_{\gamma\bar{X}}^\circ : \mathfrak{X}(\pi(M)) \longrightarrow \mathfrak{X}(\pi(M)) : \bar{Y} \longmapsto D_{\gamma\bar{X}}^\circ\bar{Y}$ is a projective invariant.*

(d) *The vector π -form $D_{\beta\bar{X}}^\circ : \mathfrak{X}(\pi(M)) \longrightarrow \mathfrak{X}(\pi(M)) : \bar{Y} \longmapsto D_{\beta\bar{X}}^\circ\bar{Y}$ is a projective invariant.*

Remark 2.8. *In view of Theorem 2.5, we conclude that, a necessary and sufficient condition for two Finsler manifolds (M, L) and (M, \tilde{L}) to be projectively related is that Relation (2.4) holds. This result generalizes the corresponding result on projective changes in Riemannian geometry. Apart from the last term of formula (2.4), this formula resembles exactly the corresponding Riemannian formula [15]. Moreover, the sufficiency is not proved before, as far as we know. On the other hand, the local expressions of (a) and (b) of Theorem 2.5 coincide with the classical local expressions [11], [14].*

For the projective change of the curvature tensors of Berwald connection, we need the following two lemmas.

Lemma 2.9. [18] *For the Berwald connection D° , we have:*

- (a) The v -curvature tensor S° vanishes.
(b) The hv -curvature tensor P° is totally symmetric.

Lemma 2.10.

- (a) The π -form $\overset{2}{D}^\circ \alpha$ is symmetric and $(\overset{2}{D}^\circ \alpha)(\bar{\eta}, \bar{X}) = 0$.
(b) The π -form $\overset{2}{D}^\circ \overset{2}{D}^\circ \alpha$ is totally symmetric and $(\overset{2}{D}^\circ \overset{2}{D}^\circ \alpha)(\bar{X}, \bar{Y}, \bar{\eta}) = -(\overset{2}{D}^\circ \alpha)(\bar{X}, \bar{Y})$.

Proof.

- (a) By definition of the Berwald vertical covariant derivative, we have

$$\begin{aligned} & (\overset{2}{D}^\circ \alpha)(\bar{X}, \bar{Y}) - (\overset{2}{D}^\circ \alpha)(\bar{Y}, \bar{X}) = (D_{\gamma\bar{X}}^\circ \alpha)(\bar{Y}) - (D_{\gamma\bar{Y}}^\circ \alpha)(\bar{X}) \\ & = \{\gamma\bar{X} \cdot (d_J \lambda(\beta\bar{Y})) - d_J \lambda(\beta D_{\gamma\bar{X}}^\circ \bar{Y})\} - \{\gamma\bar{Y} \cdot (d_J \lambda(\beta\bar{X})) - d_J \lambda(\beta D_{\gamma\bar{Y}}^\circ \bar{X})\} \\ & = \{\gamma\bar{X} \cdot (\gamma\bar{Y} \cdot \lambda) - \gamma D_{\gamma\bar{X}}^\circ \bar{Y} \cdot \lambda\} - \{\gamma\bar{Y} \cdot (\gamma\bar{X} \cdot \lambda) - \gamma D_{\gamma\bar{Y}}^\circ \bar{X} \cdot \lambda\} \\ & = \{[\gamma\bar{X}, \gamma\bar{Y}] - \gamma(D_{\gamma\bar{X}}^\circ \bar{Y} - D_{\gamma\bar{Y}}^\circ \bar{X})\} \cdot \lambda. \end{aligned}$$

From which, together with the fact that D° is torsion free, it follows that $\overset{2}{D}^\circ \alpha$ is symmetric.

On the other hand, $(\overset{2}{D}^\circ \alpha)(\bar{\eta}, \bar{X}) = 0$ is a reformulation of Lemma 2.4(b).

- (b) By (a) above and the formula

$$(\overset{2}{D}^\circ \overset{2}{D}^\circ \alpha)(\bar{X}, \bar{Y}, \bar{Z}) = \gamma\bar{X} \cdot \{(\overset{2}{D}^\circ \alpha)(\bar{Y}, \bar{Z})\} - (\overset{2}{D}^\circ \alpha)(D_{\gamma\bar{X}}^\circ \bar{Y}, \bar{Z}) - (\overset{2}{D}^\circ \alpha)(\bar{Y}, D_{\gamma\bar{X}}^\circ \bar{Z}),$$

it follows that $\overset{2}{D}^\circ \overset{2}{D}^\circ \alpha$ is symmetric with respect to the second and the third arguments and $(\overset{2}{D}^\circ \overset{2}{D}^\circ \alpha)(\bar{X}, \bar{Y}, \bar{\eta}) = -(\overset{2}{D}^\circ \alpha)(\bar{X}, \bar{Y})$. Moreover, one can show that

$$\begin{aligned} \mathfrak{A}_{\bar{X}, \bar{Y}}\{(\overset{2}{D}^\circ \overset{2}{D}^\circ \alpha)(\bar{X}, \bar{Y}, \bar{Z})\} &= \{[\gamma\bar{X}, \gamma\bar{Y}] - \gamma(D_{\gamma\bar{X}}^\circ \bar{Y} - D_{\gamma\bar{Y}}^\circ \bar{X})\} \cdot \alpha(\bar{Z}) \\ &\quad + \alpha(-D_{\gamma\bar{X}}^\circ D_{\gamma\bar{Y}}^\circ \bar{Z} + D_{\gamma\bar{Y}}^\circ D_{\gamma\bar{X}}^\circ \bar{Z} + D_{\gamma\{D_{\gamma\bar{X}}^\circ \bar{Y} - D_{\gamma\bar{Y}}^\circ \bar{X}\}}^\circ \bar{Z}) \\ &= \alpha(S^\circ(\bar{X}, \bar{Y})\bar{Z}) = 0, \quad \text{by Lemma 2.9.} \end{aligned}$$

Hence, $\overset{2}{D}^\circ \overset{2}{D}^\circ \alpha$ is symmetric with respect to the first and the second arguments. \square

Now, let us define

$$\left. \begin{aligned} Q(\bar{X}) &:= \beta\bar{X} \cdot \lambda - \lambda \alpha(\bar{X}), \\ \varepsilon(\bar{X}, \bar{Y}) &:= (D_{\gamma\bar{X}}^\circ Q)(\bar{Y}) - (D_{\gamma\bar{Y}}^\circ Q)(\bar{X}). \end{aligned} \right\} \quad (2.8)$$

Using these π -tensor fields, we have

Theorem 2.11. *Under the projective change (2.2), we have*

- (a) $\widetilde{R}^\circ(\bar{X}, \bar{Y})\bar{Z} = R^\circ(\bar{X}, \bar{Y})\bar{Z} + (D_{\gamma\bar{Z}}^\circ Q)(\bar{Y})\bar{X} - (D_{\gamma\bar{Z}}^\circ Q)(\bar{X})\bar{Y} + \varepsilon(\bar{X}, \bar{Y})\bar{Z} + (D_{\gamma\bar{Z}}^\circ \varepsilon)(\bar{X}, \bar{Y})\bar{\eta}$,

$$(b) \quad \tilde{P}^\circ(\overline{X}, \overline{Y})\overline{Z} = P^\circ(\overline{X}, \overline{Y})\overline{Z} + \mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{((\overset{2}{D}^\circ \alpha)(\overline{Y}, \overline{Z}))\overline{X}\} + (\overset{2}{D}^\circ \overset{2}{D}^\circ \alpha)(\overline{X}, \overline{Y}, \overline{Z})\overline{\eta},$$

where $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}$ denotes the cyclic sum over $\overline{X}, \overline{Y}$ and \overline{Z} .

Proof. After long, but easy, calculations, these formulae follow by using Theorem 2.5, Lemma 2.9, Lemma 2.10 and the properties of the π -forms α , Q and ε . \square

Corollary 2.12. *Under the projective change (2.2), we have*

$$(a) \quad \widetilde{R}^\circ(\overline{X}, \overline{Y}) = \widehat{R}^\circ(\overline{X}, \overline{Y}) + Q(\overline{Y})\overline{X} - Q(\overline{X})\overline{Y} + \varepsilon(\overline{X}, \overline{Y})\overline{\eta},$$

$$(b) \quad \widetilde{H}(\overline{X}) = H(\overline{X}) - Q(\overline{\eta})\overline{X} + \{Q(\overline{X}) + \varepsilon(\overline{\eta}, \overline{X})\}\overline{\eta},$$

H being the deviation tensor defined by $H(\overline{X}) := \widehat{R}^\circ(\overline{\eta}, \overline{X})$.

Proposition 2.13. *Under the projective change (2.2), if the factor of projectivity λ has the property that $Q = 0$, then the following geometric objects are projective invariants:*

(a) *The deviation tensor H ,*

(b) *The (v)h-torsion tensor \widehat{R}° ,*

(c) *The (h)h-curvature tensor R° ,*

(d) *The curvature tensor \mathfrak{R} of Barthel connection.*

Proof. The proof follows from Theorem 2.11(a), Corollary 2.12, together with the definition of H and the identity [18]

$$\mathfrak{R}(\beta\overline{X}, \beta\overline{Y}) = -\gamma\widehat{R}^\circ(\overline{X}, \overline{Y}). \quad \square$$

3. Weyl projective tensor

Studying invariant geometric objects under a given change is of particular importance. In this section, we investigate intrinsically the most important invariant tensor fields under a projective change, namely, the projective deviation tensor, the Weyl torsion tensor and the Weyl curvature tensor. The properties of these tensors and their interrelationships are investigated.

In what follows and throughout, we make use the following convention. If A is a vector π -form of degree 3, for example, we shall write $Tr_{\overline{Z}}^c\{A(\overline{X}, \overline{Y}, \overline{Z})\}$ to denote the contracted trace [3] of A with respect to \overline{Z} : $Tr_{\overline{Z}}^c\{A(\overline{X}, \overline{Y}, \overline{Z})\} := Tr^c\{\overline{Z} \mapsto A(\overline{X}, \overline{Y}, \overline{Z})\}$.

It is to be noted that if a π -tensor field A of type (1,p) is projectively invariant, then so is its contracted trace $Tr^c(A)$.

Now, let us define

$$\begin{aligned} \theta(\overline{X}, \overline{Y}) &:= Tr_{\overline{Z}}^c\{R^\circ(\overline{X}, \overline{Y})\overline{Z}\}, \\ R_2(\overline{X}, \overline{Y}) &:= Tr_{\overline{Z}}^c\{R^\circ(\overline{X}, \overline{Z})\overline{Y}\}, \\ R_1(\overline{X}) &:= \frac{1}{n-1}\{nR_2(\overline{X}, \overline{\eta}) + R_2(\overline{\eta}, \overline{X})\}; \quad n > 2, \end{aligned} \quad (3.1)$$

$$k := \frac{1}{n-1}R_2(\overline{\eta}, \overline{\eta}); \quad n > 2. \quad (3.2)$$

One can show, by using the identity [18] $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{R^\circ(\overline{X}, \overline{Y})\overline{Z}\} = 0$, that

$$\theta(\overline{X}, \overline{Y}) = R_2(\overline{X}, \overline{Y}) - R_2(\overline{Y}, \overline{X}) \quad (3.3)$$

The following lemma will be useful for subsequent use.

Lemma 3.1. *A π -tensor field ω is positively homogenous of degree r in the directional argument y (denoted by $h(r)$) if, and only if*

$$D_{\gamma\overline{\eta}}^\circ\omega = r\omega, \text{ or equivalently } D_C^\circ\omega = r\omega.$$

In view of the above lemma, we have :

Proposition 3.2.

- (a) *The hv-curvature tensor P° is homogenous of degree -1.*
- (b) *The h-curvature tensor R° is homogenous of degree 0.*
- (c) *The (v)h-torsion tensor \widehat{R}° is homogenous of degree 1.*
- (d) *The deviation tensor H is homogenous of degree 2.*
- (e) *The π -tensor fields R_2 and θ are homogenous of degree 0.*
- (f) *The π -tensor field R_1 is homogenous of degree 1.*
- (g) *The scalar function k is homogenous of degree 2.*

Now, we are in a position to announce the main result of this section.

Theorem 3.3. *Under the projective change (2.2), the following tensor field on $\pi^{-1}(TM)$; $\dim M > 2$, is projectively invariant:*

$$\begin{aligned} W(\overline{X}, \overline{Y})\overline{Z} := & R^\circ(\overline{X}, \overline{Y})\overline{Z} + \frac{1}{n+1}\mathfrak{U}_{\overline{X}, \overline{Y}}\{(D^2 R_1)(\overline{Z}, \overline{Y})\overline{X} \\ & + (D^2 R_1)(\overline{X}, \overline{Y})\overline{Z} + (D^2 D^\circ R_1)(\overline{Z}, \overline{X}, \overline{Y})\overline{\eta}\}, \end{aligned}$$

where $\mathfrak{U}_{\overline{X}, \overline{Y}}\{A(\overline{X}, \overline{Y})\} := A(\overline{X}, \overline{Y}) - A(\overline{Y}, \overline{X})$.

Proof. We have, by Corollary 2.12(a),

$$\widetilde{R}^\circ(\overline{X}, \overline{Y})\overline{\eta} = R^\circ(\overline{X}, \overline{Y})\overline{\eta} + Q(\overline{Y})\overline{X} - Q(\overline{X})\overline{Y} + \varepsilon(\overline{X}, \overline{Y})\overline{\eta}. \quad (3.4)$$

Taking the contracted trace of (3.4) with respect to \overline{Y} , we get

$$\widetilde{R}_2(\overline{X}, \overline{\eta}) = R_2(\overline{X}, \overline{\eta}) - (n-1)Q(\overline{X}) + \varepsilon(\overline{X}, \overline{\eta}). \quad (3.5)$$

On the other hand, by Theorem 2.11(a),

$$\widetilde{R}^\circ(\overline{X}, \overline{Y})\overline{Z} = R^\circ(\overline{X}, \overline{Y})\overline{Z} + (D_{\gamma\overline{Z}}^\circ Q)(\overline{Y})\overline{X} - (D_{\gamma\overline{Z}}^\circ Q)(\overline{X})\overline{Y} + \varepsilon(\overline{X}, \overline{Y})\overline{Z} + (D_{\gamma\overline{Z}}^\circ \varepsilon)(\overline{X}, \overline{Y})\overline{\eta}. \quad (3.6)$$

Taking the contracted trace of (3.6) with respect to \overline{Z} , we obtain

$$\widetilde{\theta}(\overline{X}, \overline{Y}) = \theta(\overline{X}, \overline{Y}) + (D_{\gamma\overline{X}}^\circ Q)(\overline{Y}) - (D_{\gamma\overline{Y}}^\circ Q)(\overline{X}) + n\varepsilon(\overline{X}, \overline{Y}) + (D_{\gamma\overline{\eta}}^\circ \varepsilon)(\overline{X}, \overline{Y}).$$

Since the π -form ε is $h(0)$, as one can easily show, the above relation reduces to

$$\tilde{\theta}(\bar{X}, \bar{Y}) = \theta(\bar{X}, \bar{Y}) + (n+1)\varepsilon(\bar{X}, \bar{Y}).$$

Consequently, by (3.3),

$$\varepsilon(\bar{X}, \bar{Y}) = \frac{1}{(n+1)}\{\tilde{R}_2(\bar{X}, \bar{Y}) - \tilde{R}_2(\bar{Y}, \bar{X})\} - \frac{1}{(n+1)}\{R_2(\bar{X}, \bar{Y}) - R_2(\bar{Y}, \bar{X})\}.$$

From which,

$$\varepsilon(\bar{X}, \bar{\eta}) = \frac{1}{(n+1)}\{\tilde{R}_2(\bar{X}, \bar{\eta}) - \tilde{R}_2(\bar{\eta}, \bar{X})\} - \frac{1}{(n+1)}\{R_2(\bar{X}, \bar{\eta}) - R_2(\bar{\eta}, \bar{X})\}. \quad (3.7)$$

Solving (3.5) and (3.7) for Q , taking (3.1) into account, we obtain

$$Q(\bar{X}) = \frac{1}{(n+1)}\{R_1(\bar{X}) - \tilde{R}_1(\bar{X})\}. \quad (3.8)$$

This equation, together with (2.8), yield

$$\varepsilon(\bar{X}, \bar{Y}) = \frac{1}{(n+1)}\mathfrak{U}_{\bar{X}, \bar{Y}}\{(D_{\gamma\bar{X}}^\circ R_1)(\bar{Y}) - (\tilde{D}_{\gamma\bar{X}}^\circ \tilde{R}_1)(\bar{Y})\}. \quad (3.9)$$

Substituting (3.8) and (3.9) into (3.4), we get

$$\begin{aligned} \tilde{\widehat{R}}^\circ(\bar{X}, \bar{Y}) + \frac{1}{n+1}\mathfrak{U}_{\bar{X}, \bar{Y}}\left\{\tilde{R}_1(\bar{Y})\bar{X} + (\tilde{D}_{\gamma\bar{X}}^\circ \tilde{R}_1)(\bar{Y})\bar{\eta}\right\} &= \\ = \widehat{R}^\circ(\bar{X}, \bar{Y}) + \frac{1}{n+1}\mathfrak{U}_{\bar{X}, \bar{Y}}\left\{R_1(\bar{Y})\bar{X} + (D_{\gamma\bar{X}}^\circ R_1)(\bar{Y})\bar{\eta}\right\}. & \end{aligned} \quad (3.10)$$

Now, taking the vertical covariant derivative of both sides of (3.10) with respect to \bar{Z} , making use of Corollary 2.7 and the identity [18]

$$R^\circ(\bar{X}, \bar{Y})\bar{Z} = (D_{\gamma\bar{Z}}^\circ \widehat{R}^\circ)(\bar{X}, \bar{Y}), \quad (3.11)$$

we get $\widetilde{W} = W$. □

Definition 3.4. *The π -tensor field W , defined by Theorem 3.3, is called the Weyl curvature tensor.*

In the course of the above proof, we have constructed two other important projectively invariant tensors as given by

Theorem 3.5. *Under the projective change (2.2), the following tensor fields on $\pi^{-1}(TM)$; $\dim M > 2$, are projectively invariants:*

(a) $W_1(\bar{X}) := H(\bar{X}) - k\bar{X} + \frac{1}{n+1}\{3R_1(\bar{X}) - (n+1)D_{\gamma\bar{X}}^\circ k\}\bar{\eta}$.

This tensor field is called the projective deviation tensor.

(b) $W_2(\bar{X}, \bar{Y}) := \widehat{R}^\circ(\bar{X}, \bar{Y}) + \frac{1}{n+1}\mathfrak{U}_{\bar{X}, \bar{Y}}\{R_1(\bar{Y})\bar{X} + (D_{\gamma\bar{X}}^\circ R_1)(\bar{Y})\bar{\eta}\}$.

This tensor field is called the Weyl torsion tensor.

Proof.

(a) Setting $\overline{X} = \overline{\eta}$ into (3.10), we get

$$\begin{aligned} & \tilde{H}(\overline{Y}) + \frac{1}{n+1} \left\{ \tilde{R}_1(\overline{Y})\overline{\eta} - \tilde{R}_1(\overline{\eta})\overline{Y} + (\tilde{D}_{\overline{\eta}}^\circ \tilde{R}_1)(\overline{Y})\overline{\eta} - (\tilde{D}_{\overline{Y}}^\circ \tilde{R}_1)(\overline{\eta})\overline{\eta} \right\} = \\ & = H(\overline{Y}) + \frac{1}{n+1} \left\{ R_1(\overline{Y})\overline{\eta} - R_1(\overline{\eta})\overline{Y} + (D_{\overline{\eta}}^\circ R_1)(\overline{Y})\overline{\eta} - (D_{\overline{Y}}^\circ R_1)(\overline{\eta})\overline{\eta} \right\}. \end{aligned}$$

From which, together with Proposition 3.2(f) and the identity $R_1(\overline{\eta}) = (n+1)k$ (by (3.1) and (3.2)), the result follows.

(b) Follows from (3.10). □

The next results give some interesting properties of the above mentioned projectively invariant tensors.

Theorem 3.6.

(a) *The Weyl torsion tensor W_2 can be expressed in terms of W_1 in the form*

$$W_2(\overline{X}, \overline{Y}) = \frac{1}{3} \{ (D_{\overline{X}}^\circ W_1)(\overline{Y}) - (D_{\overline{Y}}^\circ W_1)(\overline{X}) \}.$$

(b) *The Weyl curvature tensor W can be expressed in terms of W_2 in the form*

$$W(\overline{X}, \overline{Y})\overline{Z} = (D_{\overline{Z}}^\circ W_2)(\overline{X}, \overline{Y}).$$

Proof.

(a) Follows from Theorem 3.5 together with the identity [18]

$$\widehat{R}^\circ(\overline{X}, \overline{Y}) = \frac{1}{3} \{ (D_{\overline{X}}^\circ H)(\overline{Y}) - (D_{\overline{Y}}^\circ H)(\overline{X}) \},$$

taking into account the fact that $(\overset{2}{D}^\circ \overset{2}{D}^\circ k)(\overline{X}, \overline{Y}) = (\overset{2}{D}^\circ \overset{2}{D}^\circ k)(\overline{Y}, \overline{X})$.

(b) Follows from Theorem 3.5, Theorem 3.3 and (3.11). □

Corollary 3.7.

(a) *The projective deviation tensor W_1 is $h(2)$ and has the property that*

$$W_1(\overline{\eta}) = 0.$$

(b) *The Weyl torsion tensor W_2 is $h(1)$ and has the property that*

$$W_2(\overline{\eta}, \overline{X}) = W_1(\overline{X}).$$

(c) *The Weyl curvature tensor W is $h(0)$ and has the property that*

$$W(\overline{X}, \overline{Y})\overline{\eta} = W_2(\overline{X}, \overline{Y}).$$

Proof.

(a) Follows from the homogeneity properties of H , R_1 and k (Proposition 3.2) together with the fact that $R_1(\bar{\eta}) = (n+1)k$ and $H(\bar{\eta}) = 0$.

(b) Follows from (a) and Theorem 3.6(a).

(c) Follows from (b) and Theorem 3.6(b). \square

Corollary 3.8. *The following assertions are equivalent:*

(a) *The projective deviation tensor W_1 vanishes.*

(b) *The Weyl torsion tensor W_2 vanishes.*

(c) *The Weyl curvature tensor W vanishes.*

4. Projective connections and Douglas tensor

In this section, we provide a characterization of a linear connection which is invariant under a projective change (the projective connection). Moreover, as in the previous section, we investigate intrinsically another fundamental projectively invariant tensor (the Douglas tensor). Finally, we relate the Douglas tensor to the projective connection in a natural manner.

Definition 4.1. *A linear connection on $\pi^{-1}(TM)$ is said to be projective if it is invariant under the projective change (2.2).*

Theorem 4.2. *A linear connection Ω on $\pi^{-1}(TM)$ is projective if, and only if, it can be expressed in the form*

$$\Omega_X \bar{Y} = D_X^\circ \bar{Y} - \frac{1}{n+1} \{ \omega(\bar{Y}) \rho X + \omega(\rho X) \bar{Y} + (p(\rho X, \bar{Y})) \bar{\eta} \}, \quad (4.1)$$

where $\omega(\bar{Y}) := \text{Tr}^c_X \{ D_X^\circ \bar{Y} \}$ and $p(\bar{X}, \bar{Y}) := \text{Tr}^c_{\bar{Z}} \{ P^\circ(\bar{X}, \bar{Y}) \bar{Z} \}$.

Proof. Under a projective change, we have, by Theorem 2.5,

$$\tilde{D}_X^\circ \bar{Y} = D_X^\circ \bar{Y} + \alpha(\bar{Y}) \rho X + \alpha(\rho X) \bar{Y} + (\overset{2}{D}^\circ \alpha)(\rho X, \bar{Y}) \bar{\eta}, \quad (4.2)$$

Taking the contracted trace of Equation (4.2) with respect to X , using Lemma 2.10, we obtain

$$\tilde{\omega} = \omega + (n+1)\alpha,$$

from which

$$\alpha = \frac{1}{n+1} \{ \tilde{\omega} - \omega \}. \quad (4.3)$$

On the other hand, by Theorem 2.11,

$$\tilde{P}^\circ(\bar{X}, \bar{Y}) \bar{Z} = P^\circ(\bar{X}, \bar{Y}) \bar{Z} + \mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}} \{ ((\overset{2}{D}^\circ \alpha)(\bar{Y}, \bar{Z})) \bar{X} \} + (\overset{2}{D}^\circ \overset{2}{D}^\circ \alpha)(\bar{X}, \bar{Y}, \bar{Z}) \bar{\eta} \quad (4.4)$$

Taking the contracted trace of the above equation with respect to \bar{Z} , using Lemma 2.10, we get

$$\tilde{p} = p + (n+1) \overset{2}{D}^\circ \alpha,$$

from which

$$\overset{2}{D}^\circ \alpha = \frac{1}{(n+1)} \{\tilde{p} - p\}. \quad (4.5)$$

Substituting (4.3) and (4.5) into (4.2), the result follows.

It should finally be noted that the projective connection Ω is uniquely determined by Equation (4.1). \square

Remark 4.3. *In view of (4.5), $\overset{2}{D}^\circ \alpha = 0$ if, and only if, the π -form p is projectively invariant. In this case, the formula (2.4) reduces to*

$$\tilde{D}_X^\circ \bar{Y} = D_X^\circ \bar{Y} + \alpha(\bar{Y})\rho X + \alpha(\rho X)\bar{Y},$$

which has exactly the same form as the corresponding Riemannian formula for projective changes.

Proposition 4.4. *The projective connection Ω has the properties:*

- (a) *The (classical) torsion associated with Ω vanishes.*
- (b) *The v -curvature tensor associated with Ω vanishes.*

Proof.

(a) As P° is totally symmetric (Lemma 2.9(b)), then so is the π -form p . Then, the result follows from this fact and the expression (4.1).

(b) For any regular connection D having the property that $T(\bar{X}, \bar{\eta}) = 0$, the v -curvature tensor S of D takes the form [18]:

$$\begin{aligned} S(\bar{X}, \bar{Y})\bar{Z} &= (D_{\bar{Y}}T)(\bar{X}, \bar{Z}) - (D_{\bar{X}}T)(\bar{Y}, \bar{Z}) + T(\bar{X}, T(\bar{Y}, \bar{Z})) \\ &\quad - T(\bar{Y}, T(\bar{X}, \bar{Z})) + T(\hat{S}(\bar{X}, \bar{Y}), \bar{Z}). \end{aligned}$$

The result follows from the above relation together with (a). \square

Theorem 4.5. *Under a projective change, the π -tensor field*

$$\mathbb{P}(\bar{X}, \bar{Y})\bar{Z} := P^\circ(\bar{X}, \bar{Y})\bar{Z} - \frac{1}{n+1} \mathfrak{G}_{\bar{X}, \bar{Y}, \bar{Z}} \{ (p(\bar{X}, \bar{Y}))\bar{Z} \} - \frac{1}{n+1} \{ (D_{\bar{Y}}^\circ p)(\bar{X}, \bar{Z}) \} \bar{\eta}. \quad (4.6)$$

is invariant.

Proof. From Equation (4.5) and Corollary 2.7(c), we have

$$\overset{2}{D}^\circ \alpha = \frac{1}{(n+1)} \{\tilde{p} - p\}, \quad \overset{2}{D}^\circ \overset{2}{D}^\circ \alpha = \frac{1}{(n+1)} \{ \tilde{\overset{2}{D}}^\circ \tilde{p} - \overset{2}{D}^\circ p \}.$$

From which, together with Theorem 2.11(b), the result follows. \square

Definition 4.6. *The π -tensor field \mathbb{P} of type (1,3) on $\pi^{-1}(TM)$ defined by (4.6) is called the Douglas tensor associated with the projective change (2.2).*

The following result establishes some important properties of the Douglas tensor.

Proposition 4.7. *The Douglas tensor \mathbb{P} has the properties:*

- (a) *\mathbb{P} vanishes if P° vanishes,*

(b) $\widehat{\mathbb{P}}(\overline{X}, \overline{Y}) = 0,$

(c) \mathbb{P} is totally symmetric,

(d) $(D_{\gamma\overline{X}}^\circ\mathbb{P})(\overline{Y}, \overline{Z}, \overline{W}) = (D_{\gamma\overline{Z}}^\circ\mathbb{P})(\overline{Y}, \overline{X}, \overline{W}),$

(e) \mathbb{P} is positively homogenous of degree -1 in y .

Proof. The proof is easy and we omit it. \square

Remark 4.8. *It is worth noting that in projective Riemannian geometry there is only one fundamental projectively invariant tensor (Weyl curvature tensor), whereas in projective Finsler geometry there are two fundamental projectively invariant tensors, one is the Weyl curvature tensor W and the other is the Douglas tensor \mathbb{P} .*

We terminate this section by the following result which relates the Douglas tensor with the projective connection in a natural manner. This result says roughly that the Douglas tensor is completely determined by the projective connection.

Theorem 4.9. *The Douglas tensor is precisely the hv-curvature tensor of the projective connection (4.1).*

Proof. Let \overline{K} and $\overline{\beta}$ be the connection map and the horizontal map of the projective connection Ω respectively. By Theorem 4.2 and the fact that $p(\overline{X}, \overline{\eta}) = 0$, the connection map \overline{K} takes the form

$$\overline{K}(X) = K(X) - \frac{1}{n+1}\{\omega(\overline{\eta})\rho X + \omega(\rho X)\overline{\eta}\}.$$

From which, together with Proposition 4.4(a) and Proposition 2.2 of [18], the horizontal map $\overline{\beta}$ is given by

$$\overline{\beta}(\overline{X}) = \beta(\overline{X}) + \frac{1}{n+1}\{\omega(\overline{\eta})\gamma\overline{X} + \omega(\overline{X})\gamma\overline{\eta}\}. \quad (4.7)$$

Now, after somewhat long calculations, using (4.7) and Theorem 1.1, we get

$$\begin{aligned} \Omega_{\beta\overline{X}}\Omega_{\gamma\overline{Y}}\overline{Z} &= D_{\beta\overline{X}}^\circ D_{\gamma\overline{Y}}^\circ \overline{Z} - \frac{1}{n+1}\{\omega(D_{\gamma\overline{Y}}^\circ \overline{Z})\overline{X} + \omega(\overline{X})D_{\gamma\overline{Y}}^\circ \overline{Z} + (p(\overline{X}, D_{\gamma\overline{Y}}^\circ \overline{Z}))\overline{\eta}\} \\ &\quad + \frac{1}{n+1}\{\omega(\overline{\eta})D_{\gamma\overline{X}}^\circ D_{\gamma\overline{Y}}^\circ \overline{Z} + \omega(\overline{X})D_{\gamma\overline{\eta}}^\circ D_{\gamma\overline{Y}}^\circ \overline{Z}\}, \end{aligned}$$

$$\begin{aligned} \Omega_{\gamma\overline{Y}}\Omega_{\beta\overline{X}}\overline{Z} &= D_{\gamma\overline{Y}}^\circ D_{\beta\overline{X}}^\circ \overline{Z} - \frac{1}{n+1}\{(D_{\gamma\overline{Y}}^\circ \omega(\overline{X}))\overline{Z} + \omega(\overline{X})D_{\gamma\overline{Y}}^\circ \overline{Z} + \\ &\quad + (D_{\gamma\overline{Y}}^\circ \omega(\overline{Z}))\overline{X} + \omega(\overline{Z})D_{\gamma\overline{Y}}^\circ \overline{X}\} - \frac{1}{n+1}\{(D_{\gamma\overline{Y}}^\circ p(\overline{X}, \overline{Z}))\overline{\eta} + (p(\overline{X}, \overline{Z}))\overline{Y}\} \\ &\quad + \frac{1}{n+1}\{(D_{\gamma\overline{Y}}^\circ \omega(\overline{\eta}))D_{\gamma\overline{X}}^\circ \overline{Z} + (D_{\gamma\overline{Y}}^\circ \omega(\overline{X}))D_{\gamma\overline{\eta}}^\circ \overline{Z}\} \\ &\quad + \frac{1}{n+1}\{\omega(\overline{\eta})D_{\gamma\overline{Y}}^\circ D_{\gamma\overline{X}}^\circ \overline{Z} + \omega(\overline{X})D_{\gamma\overline{Y}}^\circ D_{\gamma\overline{\eta}}^\circ \overline{Z}\} \end{aligned}$$

and

$$\begin{aligned}\Omega_{[\beta\bar{X}, \gamma\bar{Y}]} \bar{Z} &= D_{[\beta\bar{X}, \gamma\bar{Y}]}^\circ \bar{Z} + \frac{1}{n+1} \{ \omega(\bar{Z}) D_{\gamma\bar{Y}}^\circ \bar{X} + \omega(D_{\gamma\bar{Y}}^\circ \bar{X}) \bar{Z} + (p(D_{\gamma\bar{Y}}^\circ \bar{X}, \bar{Z})) \bar{\eta} \} \\ &\quad + \frac{1}{n+1} \{ \omega(\bar{\eta}) D_{[\gamma\bar{X}, \gamma\bar{Y}]}^\circ \bar{Z} + \omega(\bar{X}) D_{[\gamma\bar{\eta}, \gamma\bar{X}]}^\circ \bar{Z} \} \\ &\quad - \frac{1}{n+1} \{ (D_{\gamma\bar{Y}}^\circ \omega(\bar{\eta})) D_{\gamma\bar{X}}^\circ \bar{Z} + (D_{\gamma\bar{Y}}^\circ \omega(\bar{X})) D_{\gamma\bar{\eta}}^\circ \bar{Z} \}.\end{aligned}$$

The result follows from the above three relations together with Lemma 2.9(a) and the fact that $p = D_{\gamma\bar{Y}}^\circ \omega$. \square

5. Projectively flat manifolds

In this section we investigate intrinsically the projective change of some important special Finsler manifolds, namely, the Berwald, Douglas and projectively flat Finsler manifolds. Moreover, the relationship between projectively flat Finsler manifolds and the Douglas tensor (Weyl tensor) is obtained.

Definition 5.1. *A Finsler manifold (M, L) is said to be a Berwald manifold if the $(h)hv$ -torsion tensor T of the Cartan connection ∇ is horizontally parallel. That is,*

$$\nabla_{\beta\bar{X}} T = 0.$$

Definition 5.2. *A Finsler manifold (M, L) is said to be a Douglas manifold if its Douglas tensor vanishes identically: $\mathbb{P} = 0$.*

Theorem 5.3. *Every Berwald manifold is a Douglas manifold.*

Proof. We first show that the hv -curvature P° of a Berwald manifold vanishes. The hv -curvature P and the $(v)hv$ -torsion of the Cartan connection can be put respectively in the form [18]:

$$\begin{aligned}P(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &= g((\nabla_{\beta\bar{Z}} T)(\bar{X}, \bar{Y}), \bar{W}) - g((\nabla_{\beta\bar{W}} T)(\bar{X}, \bar{Y}), \bar{Z}) \\ &\quad + g(T(\bar{X}, \bar{Z}), \hat{P}(\bar{W}, \bar{Y})) - g(T(\bar{X}, \bar{W}), \hat{P}(\bar{Z}, \bar{Y})), \\ \hat{P}(\bar{X}, \bar{Y}) &= (\nabla_{\beta\bar{X}} T)(\bar{X}, \bar{Y}).\end{aligned}$$

But since $\nabla_{\beta\bar{X}} T = 0$, then both \hat{P} and P vanish.

On the other hand, the hv -curvatures P° and P are related by [18]:

$$\begin{aligned}P^\circ(\bar{X}, \bar{Y}) \bar{Z} &= P(\bar{X}, \bar{Y}) \bar{Z} + (\nabla_{\gamma\bar{Y}} \hat{P})(\bar{X}, \bar{Z}) + \hat{P}(T(\bar{Y}, \bar{X}), \bar{Z}) + \hat{P}(\bar{X}, T(\bar{Y}, \bar{Z})) \\ &\quad + (\nabla_{\beta\bar{X}} T)(\bar{Y}, \bar{Z}) - T(\bar{Y}, \hat{P}(\bar{X}, \bar{Z})) - T(\hat{P}(\bar{X}, \bar{Y}), \bar{Z}).\end{aligned}$$

From which, together with $\nabla_{\beta\bar{X}} T = \hat{P} = P = 0$, it follows that $P^\circ = 0$.

Now, the vanishing of P° implies that $p = 0$. Hence, by Theorem 4.5, the Douglas tensor \mathbb{P} vanishes identically. \square

As a consequence of Theorem 5.3, we retrieve intrinsically a result of Matsumoto [9]:

Corollary 5.4. *A Finsler manifold which is projective to a Berwald manifold is a Douglas manifold.*

Proof. Since (M, \widetilde{L}) is a Berwald manifold, then, its hv -curvature \widetilde{P}° vanishes and consequently $\widetilde{p} = 0$. Hence, the Douglas tensor \mathbb{P} , which coincides with $\widetilde{\mathbb{P}}$ (by Theorem 4.5), vanishes identically. \square

Now, we focus our attention on projectively flat Finsler manifolds.

Definition 5.5. *Under a projective change, a Finsler manifold (M, L) is said to be:*

- *hv -projectively flat if the hv -curvature tensor \widetilde{P}° vanishes.*
- *h -projectively flat if the h -curvature tensor \widetilde{R}° vanishes.*
- *projectively flat if both \widetilde{P}° and \widetilde{R}° vanish.*

In view of the proof of Corollary 5.4 and Definition 5.5, we have

Corollary 5.6. *Under the projective change (2.2), if (M, L) is hv -projectively flat, then it is a Douglas manifold.*

The converse is also true under a certain condition: Consider a projective change for which the projective factor λ (or α) satisfies the condition

$$D^\circ \alpha = -\frac{1}{(n+1)} p. \quad (5.1)$$

From which, we get

$$D^\circ D^\circ \alpha = -\frac{1}{(n+1)} D^\circ p. \quad (5.2)$$

On the other hand, if the Douglas tensor \mathbb{P} vanishes, then, by Theorem 4.5, we obtain

$$P^\circ(\overline{X}, \overline{Y})\overline{Z} = \frac{1}{n+1} \mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{ (p(\overline{X}, \overline{Y}))\overline{Z} \} + \frac{1}{n+1} \{ (D^\circ p)(\overline{Y}, \overline{X}, \overline{Z}) \} \overline{\eta}.$$

From which, together with (4.4) and Lemma 2.10, the hv -curvature \widetilde{P}° has the form

$$\begin{aligned} \widetilde{P}^\circ(\overline{X}, \overline{Y})\overline{Z} &= \frac{1}{n+1} \mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{ (p(\overline{X}, \overline{Y}))\overline{Z} \} + \frac{1}{n+1} \{ (D^\circ p)(\overline{Y}, \overline{X}, \overline{Z}) \} \overline{\eta} \\ &\quad + \mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{ ((D^\circ \alpha)(\overline{Y}, \overline{Z}))\overline{X} \} + (D^\circ D^\circ \alpha)(\overline{X}, \overline{Y}, \overline{Z}) \overline{\eta} \\ &= \mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{ \{ (D^\circ \alpha)(\overline{Y}, \overline{Z}) + \frac{1}{(n+1)} p(\overline{Y}, \overline{Z}) \} \overline{X} \} \\ &\quad \{ (D^\circ D^\circ \alpha)(\overline{Y}, \overline{X}, \overline{Z}) + \frac{1}{(n+1)} (D^\circ p)(\overline{Y}, \overline{X}, \overline{Z}) \} \overline{\eta}. \end{aligned}$$

Now, using (5.1) and (5.2), \widetilde{P}° vanishes. Hence, we have

Theorem 5.7. *Under a projective change satisfying (5.1), a Finsler manifold (M, L) is hv -projectively flat if, and only if, (M, L) is a Douglas manifold.*

Now, we study h -projectively flat Finsler manifolds

Proposition 5.8. *Under the projective change (2.2), if (M, L) ; $\dim M > 2$, is h -projectively flat, then its Weyl torsion tensor vanishes .*

Proof. Since (M, L) is h -projectively flat, then, by Definition 5.5, \widetilde{R}° vanishes and hence $\widetilde{R}_2 = \widetilde{R}_1 = 0$. Consequently, by Theorem 3.5(b), \widetilde{W}_2 vanishes. As the Weyl torsion tensor W_2 is invariant, then W_2 also vanishes. \square

The converse is also true under a certain condition: Consider a projective change for which the projective factor λ (or α) satisfies the condition

$$Q(\overline{X}) = \frac{1}{(n+1)}R_1(\overline{X}). \quad (5.3)$$

This condition implies that

$$\varepsilon(\overline{X}, \overline{Y}) = \frac{1}{(n+1)}\{(D_{\overline{\gamma X}}^\circ R_1)(\overline{Y}) - (D_{\overline{\gamma Y}}^\circ R_1)(\overline{X})\}. \quad (5.4)$$

On the other hand, if the Weyl torsion tensor W_2 vanishes, then, by Theorem 3.5(b), we obtain

$$\widehat{R}^\circ(\overline{X}, \overline{Y}) = -\frac{1}{n+1}\mathfrak{U}_{\overline{X}, \overline{Y}}\left\{R_1(\overline{Y})\overline{X} + (D_{\overline{\gamma X}}^\circ R_1)(\overline{Y})\overline{\eta}\right\}.$$

From which, together with (3.4), we have

$$\begin{aligned} \widetilde{R}^\circ(\overline{X}, \overline{Y}) &= -\frac{1}{n+1}\mathfrak{U}_{\overline{X}, \overline{Y}}\left\{R_1(\overline{Y})\overline{X} + (D_{\overline{\gamma X}}^\circ R_1)(\overline{Y})\overline{\eta}\right\} \\ &\quad + Q(\overline{Y})\overline{X} - Q(\overline{X})\overline{Y} + \varepsilon(\overline{X}, \overline{Y})\overline{\eta} \\ &= \left\{Q(\overline{Y}) - \frac{1}{(n+1)}R_1(\overline{Y})\right\}\overline{X} - \left\{Q(\overline{X}) - \frac{1}{(n+1)}R_1(\overline{X})\right\}\overline{Y} \\ &\quad + \left\{\varepsilon(\overline{X}, \overline{Y}) - \frac{1}{(n+1)}\{(D_{\overline{\gamma X}}^\circ R_1)(\overline{Y}) - (D_{\overline{\gamma Y}}^\circ R_1)(\overline{X})\}\right\}\overline{\eta}. \end{aligned}$$

Now, using (5.3) and (5.4), \widetilde{R}° vanishes. Consequently, \widehat{R}° vanishes, by (3.11). Hence, we have

Theorem 5.9. *Under a projective change satisfying (5.3), a Finsler manifold (M, L) ; $\dim M > 2$, is h -projectively flat if, and only if, its Weyl torsion tensor vanishes.*

Combining Theorem 5.7 and Theorem 5.9, we retrieve intrinsically a result of Matsumoto [9]. Namely, we have

Theorem 5.10. *Under a projective change satisfying (5.1) and (5.3), a Finsler manifold (M, L) ; $\dim M > 2$, is projectively flat if, and only if, its Weyl torsion tensor and Douglas tensor vanish.*

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